

Formal matched asymptotics for degenerate Ricci flow neckpinches

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ABSTRACT. Gu and Zhu [16] have shown that Type-II Ricci flow singularities develop from nongeneric rotationally symmetric Riemannian metrics on S^{n+1} ($n \geq 2$). In this paper, we describe and provide plausibility arguments for a detailed asymptotic profile and rate of curvature blow-up that we predict such solutions exhibit.

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1. Introduction

Let $(S^{n+1}, g(t) : 0 \leq t < T)$ be a rotationally symmetric solution of Ricci flow. Gu and Zhu prove that Type-II singularities can occur for special nongeneric initial data of this type [16]. Garfinkle and one of the authors provide numerical simulations of the formation of such singularities at either pole [13, 14]. However, almost nothing is known about the asymptotics of such singularity formation, except for complete noncompact solutions on \mathbb{R}^2 , where Ricci flow coincides with logarithmic fast diffusion, $u_t = \Delta \log u$. (The asymptotics of logarithmic fast diffusion, which are unrelated to the results in this paper, were derived by King [18] and subsequently proved in \mathbb{R}^2 by Daskalopoulos and Šešum [10].)

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We say a sequence $\{(p_i, t_i)\}_{i=0}^\infty$ of points and times in a Ricci flow solution is a *blow-up sequence* at time T if $t_i \nearrow T$ and $|\text{Rm}(x_i, t_i)| \rightarrow \infty$ as $i \rightarrow \infty$. We say $(\mathcal{M}^{n+1}, g(t))$ develops a *neckpinch singularity* at $T < \infty$ if there is some blow-up sequence at T whose corresponding sequence of parabolic dilations has as a pointed limit the self-similar Ricci soliton on the cylinder $\mathbb{R} \times \mathbb{S}^n$. We call a neckpinch *nondegenerate* if every complete pointed limit formed from a blowup sequence at T is a solution on the cylinder; we call it *degenerate* if a complete smooth limit of some blow-up sequence has another topology. Rotationally symmetric nondegenerate neckpinches have been studied by Simon [21] and two of the authors [2, 3], who proved they are Type-I (“rapidly forming”) singularities in which the curvature blows up at the natural parabolic rate, $(T - t) \sup_{p \in \mathcal{M}^{n+1}} |\text{Rm}(p, t)| < \infty$. On the other hand, Type-II (“slowly forming”) singularities have the property that $(T - t) \sup_{p \in \mathcal{M}^{n+1}} |\text{Rm}(p, t)| = \infty$. The compact Type-II singularities proved to exist by Gu and Zhu [16] are degenerate neckpinches.

This paper is the first of two in which we study the formation of degenerate Ricci flow neckpinch singularities. In the present work, we assume that a degenerate neckpinch singularity occurs at a pole at time $T < \infty$, and we derive formal matched asymptotics for the solution as it approaches the singularity. This procedure provides evidence for a conjectural picture of the behavior of some (not necessarily all) solutions that develop rotationally symmetric degenerate neckpinch singularities. In particular, it predicts precise rates of Type-II curvature blow-up.¹ In forthcoming work, we will provide rigorous proof that there exist solutions exhibiting the asymptotic behavior formally described here.

Given any rotationally symmetric family $g(t)$ of metrics on \mathbb{S}^{n+1} , one may remove the poles P_\pm and write the metrics on $\mathbb{S}^{n+1} \setminus \{P_\pm\} \approx (-1, 1) \times \mathbb{S}^n$ in the form $g(t) = (ds)^2 + \psi^2(s, t) g_{\text{can}}$, where $s(x, t)$ denotes $g(t)$ -arclength to $x \in [-1, 1]$ from a fixed point $x_0 \in (-1, 1)$. (Here, g_{can} is the canonical unit sphere metric on \mathbb{S}^n ; see Section 2 for a detailed discussion of these coordinates.) Thus the function $\psi(s(x, t), t)$ completely characterizes a given solution. Our Basic Assumption in this paper, explained in detail in Section 2, is that the initial metric $g(s, 0)$, hence $\psi(s, 0)$, satisfies certain curvature restrictions which ensure that the geometries we consider are sufficiently close to those studied in [13, 14] and [16], and therefore are likely to develop neckpinches. It also allows us to employ certain prior results of two of the authors [2], which are useful for the arguments made here. (Compare [3].)

In the language of Section 2, let $\hat{s}(t)$ denote the location of the local maximum (“bump”) of ψ closest to the right pole $x = 1$. Note that $\hat{s}(t)$ may be only an upper-semicontinuous function of time, because a bump and an adjacent neck can join and annihilate each other. Lemma 7.1 of [2] proves that the solution exists until ψ becomes zero somewhere other than at the poles. Since there is always at least one positive local maximum of ψ , the quantity $\hat{s}(t)$ is defined for as long as the solution exists. Lemma 7.2 of [2] proves that if $\lim_{t \nearrow T} \psi(\hat{s}(t), t) > 0$, then no singularity occurs at the right pole. Hence, if a degenerate neckpinch does happen at the right pole, it must be that $\lim_{t \nearrow T} \psi(\hat{s}(t), t) = 0$. This can happen only if (i) the radius ψ vanishes on an open set, or (ii) the bump marked by \hat{s} moves to the right pole. Note that these alternatives are not mutually exclusive. In either case,

¹For a comprehensive statement of our results, see Section 7.

it follows that there are points $\tilde{s}(t)$ such that

$$(1.1) \quad \lim_{t \nearrow T} \tilde{s}(t) = s(1, T) \quad \text{and} \quad \lim_{t \nearrow T} \psi_s(\tilde{s}(t), t) = 0.$$

Observe that (1.1) is incompatible with the boundary condition (2.3) necessary for regularity at the pole, which forces $\psi = (s(1) - s) + o(s(1) - s)$ as $s \nearrow s(1)$. Therefore, a consequence of our Basic Assumption is that if a degenerate singularity does develop, then this expansion for ψ cannot hold uniformly in s as $t \nearrow T$. Instead, the solution must exhibit different qualitative behaviors in a sequence of time-dependent spatial regions.

Starting with that fact, we construct in this paper a conjectural model for rotationally symmetric Ricci flow solutions that develop a degenerate neckpinch, and we check its consistency. We do this systematically by studying approximate asymptotic expansions to the solution in four connected regions, which we call (moving in from the pole) the *tip*, *parabolic*, *intermediate*, and *outer* regions. By matching these expansions at the intersections of the sequential regions, we produce our model. This is the essence of the formal matched asymptotics process familiar to applied mathematicians. The process involves first formulating an *Ansatz*, which consists of a series of assumptions (listed below) pertaining to the geometry and its evolution equations in the four regions. Then, matching across the boundaries of the regions, one constructs formal (approximate) solutions. Finally, one checks that these formal solutions remain consistent with the *Ansatz*, and argues that the approximations remain sufficiently accurate.

For clarity of exposition, we establish notation in Section 2 and then begin our study working in the parabolic region rather than the tip. We treat the tip (the most critical region) last. Finally, in Section 7, we provide a detailed summary of our results and the conjectural picture they provide.

2. Basic equations

We begin by recalling some basic identities for $\text{SO}(n+1)$ -invariant Ricci flow solutions.

To avoid working in multiple patches, it is convenient to puncture the sphere \mathbb{S}^{n+1} at the poles P_{\pm} and identify $\mathbb{S}^{n+1} \setminus \{P_{\pm}\}$ with $(-1, 1) \times \mathbb{S}^n$. If we let x denote the coordinate on the interval $(-1, 1)$ and let g_{can} denote the canonical unit sphere metric, then an arbitrary family $g(t)$ of smooth $\text{SO}(n+1)$ -invariant metrics on \mathbb{S}^{n+1} may be written in geodesic polar coordinates as

$$(2.1) \quad g(t) = \varphi^2(x, t) (dx)^2 + \psi^2(x, t) g_{\text{can}}.$$

Denoting the distance from the equator $\{0\} \times \mathbb{S}^n$ by

$$s(x, t) = \int_0^x \varphi(\xi, t) d\xi,$$

allows one to write (2.1) in the more geometrically natural form

$$(2.2) \quad g = (ds)^2 + \psi^2(s(x, t), t) g_{\text{can}},$$

where “d” is the differential with respect to the space variables but not the time variables. We also agree to write

$$\frac{\partial}{\partial s} = \frac{1}{\varphi(x, t)} \frac{\partial}{\partial x}.$$

With this convention, $\partial/\partial s$ and $\partial/\partial t$ do not commute; instead one gets the usual commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -\frac{\varphi_t}{\varphi} \frac{\partial}{\partial s}.$$

Smoothness at the poles requires that ψ satisfy the boundary conditions

$$(2.3) \quad \psi_s|_{x=\pm 1} = \mp 1.$$

The metric (2.2) has two distinguished sectional curvatures: the curvature

$$(2.4) \quad L = \frac{1 - \psi_s^2}{\psi^2}$$

of a plane tangent to $\{s\} \times \mathbb{S}^n$, and the curvature

$$(2.5) \quad K = -\frac{\psi_{ss}}{\psi}$$

of an orthogonal plane.

Under Ricci flow,

$$\frac{\partial}{\partial t} g = -2\text{Rc},$$

the quantities φ and ψ evolve by the degenerate parabolic system

$$\begin{aligned} \varphi_t &= n \left(\frac{\psi_{xx}}{\varphi\psi} - \frac{\varphi_x\psi_x}{\varphi^2\psi} \right), \\ \psi_t &= \frac{\psi_{xx}}{\varphi^2} - \frac{\varphi_x\psi_x}{\varphi^3} + (n-1) \frac{\psi_x^2}{\varphi^2\psi} - \frac{n-1}{\psi}. \end{aligned}$$

The degeneracy is due to invariance of the system under the infinite-dimensional diffeomorphism group. By writing the spatial derivatives in terms of $\partial/\partial s$, one can simplify the appearance of these equations, effectively by fixing arclength as a gauge. The quantities φ and ψ in (2.1) then evolve by

$$(2.6a) \quad \varphi_t = n \frac{\psi_{ss}}{\psi} \varphi,$$

$$(2.6b) \quad \psi_t = \psi_{ss} - (n-1) \frac{1 - \psi_s^2}{\psi},$$

respectively.

In [2], nondegenerate neckpinch singularity formation is established for an open set of initial data of the form (2.2) on \mathbb{S}^{n+1} ($n \geq 2$) satisfying the following assumptions:²

- (1) The sectional curvature L of planes tangent to each sphere $\{s\} \times \mathbb{S}^n$ is positive.
- (2) The Ricci curvature $\text{Rc} = nK(ds)^2 + [K + (n-1)L]\psi^2 g_{\text{can}}$ is positive on each polar cap.
- (3) The scalar curvature $R = 2nK + n(n-1)L$ is positive everywhere.
- (4) The metric has at least one neck and is “sufficiently pinched” in the sense that the value of ψ at the smallest neck is sufficiently small relative to its value at either adjacent bump.

In [3], precise asymptotics are derived under the additional assumption:

²We call local minima of ψ “necks” and local maxima “bumps”. The “polar caps” are the regions on either side of the outermost bumps.

- (5) The metric is reflection symmetric, $\psi(s) = \psi(-s)$, and the smallest neck is at $x = 0$.

For our analysis here of degenerate neckpinch solutions, we start from initial geometric data that satisfy some but not all of these assumptions. In particular, we drop the condition on the “tightness” of the initial neck pinching, and we also do not require reflection symmetry.³ We now state our underlying assumptions on the initial geometry:

Basic Assumption. *The solutions of Ricci flow we consider in this paper are $\mathrm{SO}(n+1)$ -invariant. They satisfy conditions (1)–(3) above, and they have initial data with at least one neck. Furthermore, we assume that a singularity occurs at the right pole ($x = +1$) at some time $T < \infty$.*

3. The parabolic region

A consequence of our Basic Assumption is that $R > 0$ at $t = 0$. By standard arguments, this implies that the solution becomes singular at some time $T < \infty$.

We assume that a neckpinch occurs at some x_0 ; and by diffeomorphism invariance, we may assume that $x_0 = 0$. Thus in the parabolic region to be characterized below, it is natural to study the system in the renormalized variables τ and σ defined by

$$(3.1) \quad \tau(t) := -\log(T - t),$$

and

$$(3.2) \quad \sigma(x, t) := \frac{s(x, t)}{\sqrt{T - t}} = e^{\tau/2} s(x, t),$$

where $s(x, t)$ denotes arclength from the point $x = 0$. Note that σ is a natural choice for the parabolic region, in which time scales like distance squared. Use of τ is not necessary but is convenient for the calculations that follow. (Compare [3].)

The evolution equation satisfied by ψ , equation (2.6b), implies that at the local maximum (“bump”) $\hat{s}(t)$ closest to the right pole, one has $\psi_t(\hat{s}, t) \leq (n-1)/\psi$. On the other hand, applying a version of the maximum principle to the same equation shows that the radius of the smallest neck satisfies the condition $\psi_{\min}(t) \geq \sqrt{(n-1)(T-t)}$. (For the proof, see Lemma 6.1 of [2].) Hence, we introduce the rescaled radius

$$(3.3) \quad U(\sigma, \tau) := \frac{\psi(s, t)}{\sqrt{2(n-1)(T-t)}}.$$

Note that U is a function of the coordinate σ . In Appendix A, we compute the evolution equation satisfied by U , which is

$$(3.4) \quad U_\tau = U_{\sigma\sigma} - \left(\frac{\sigma}{2} + nI\right) U_\sigma + (n-1) \frac{U_\sigma^2}{U} + \frac{1}{2} \left(U - \frac{1}{U}\right),$$

where

$$I := \int_0^\sigma \frac{U_{\sigma\sigma}}{U} d\sigma.$$

Note that equation (3.4) is parabolic, but contains a nonlocal term.

³As seen in Section 3, some though not all of the solutions we construct here are reflection symmetric.

Motivated by Perelman's work [20] in dimension $n + 1 = 3$, one expects the solution to be approximately cylindrical a controlled distance away from the pole, depending only on the curvature there. (Essentially, this is contained in Corollary 11.8 of [20]; for more details and a refinement, see Sections 47–48 and in particular Corollary 48.1 of [19].) This expectation effectively determines the behavior we anticipate in the parabolic region Ω_{par} .

Ansatz Condition 1. *As $\tau \nearrow \infty$ (equivalently, as $t \nearrow T$), the rescaled radius U converges uniformly in regions $|\sigma| \leq \text{const}$, where*

$$U(\sigma, \tau) \rightarrow 1.$$

Since an exact cylinder solution is given by $U(\sigma, \tau) = 1$, the parabolic region Ω_{par} may be characterized as that within which the quantity

$$(3.5) \quad V(\sigma, \tau) := U(\sigma, \tau) - 1$$

is small.

By substituting $U = 1 + V$ in equation (3.4),⁴ one finds that V evolves by

$$(3.6) \quad V_\tau = AV + N(V),$$

where A is the linear operator

$$(3.7) \quad AV := V_{\sigma\sigma} - \frac{\sigma}{2}V_\sigma + V,$$

and N consists of nonlinear terms,

$$(3.8) \quad N(V) := \frac{2(n-1)V_\sigma^2 - V^2}{2(1+V)} - nIV_\sigma,$$

with $I = \int_0^\sigma V_{\sigma\sigma}/(1+V) d\sigma$ its sole non-local term.

The operator A appears in many blow-up problems for semilinear parabolic equations of the type $u_t = u_{xx} + u^p$ (e.g. [15]), and also in the analysis of mean curvature flow neckpinches (e.g. [4]). The operator A is self adjoint in $L^2(\mathbb{R}, e^{-\sigma^2/4} d\sigma)$. It has pure point spectrum with eigenvalues $\{\lambda_k\}_{k=0}^\infty$, where

$$(3.9) \quad \lambda_k := 1 - \frac{k}{2}.$$

The associated eigenfunctions are the Hermite polynomials $h_k(\sigma)$. We adopt the normalization that the highest-order term has coefficient 1, so that $h_0(\sigma) = 1$, $h_1(\sigma) = \sigma$, $h_2(\sigma) = \sigma^2 - 2$, $h_3(\sigma) = \sigma^3 - 6\sigma$, and in general,

$$h_k(\sigma) = \sum_{j=0}^k \eta_j \sigma^j,$$

for certain determined constants η_j , with $\eta_k = 1$. Recall that this sum for the Hermite polynomial h_k contains only odd powers of σ if k is odd, and even powers if k is even.

In the parabolic region Ω_{par} , which may be regarded as the space-time region where one expects $V \approx 0$, the linear term AV should dominate the nonlinear term $N(V)$, as long as V is orthogonal to the kernel of A . (We argue below that this orthogonality follows from the *Ansatz* adopted in this paper.) Therefore, we may

⁴See equation (A.7).

assume that $V \approx \tilde{V}$, where \tilde{V} is an exact solution of the approximate (linearized) equation $\tilde{V}_\tau = A\tilde{V}$.

The general solution to the linear PDE $\tilde{V}_\tau = A\tilde{V}$ can be written in the form of a Fourier series by expanding in the Hermite eigenfunctions, namely $\tilde{V}(\sigma, \tau) = \sum_{k=0}^{\infty} b_k e^{\lambda_k \tau} h_k(\sigma)$, where all coefficients b_k except possibly b_2 are constants. If we treat \tilde{V} as the first term in an approximate expansion for solutions of the nonlinear system (3.6), then we are led to write

$$(3.10) \quad V(\sigma, \tau) \approx b_0 e^\tau + b_1 e^{\tau/2} h_1(\sigma) + b_2(\tau) h_2(\sigma) + \sum_{k=3}^{\infty} b_k e^{\lambda_k \tau} h_k(\sigma),$$

where $b_2(\tau)$ is allowed to be a function of time, since it corresponds to a null eigenvalue of the linearized operator and hence to motion on a center manifold where the contributions of $N(V)$ cannot be ignored.⁵ In a standard initial value problem, the coefficients b_k are determined by the initial data. Here, they may be determined by matching at the intersection with the adjacent regions.

We now make a pair of additional assumptions.

Ansatz Condition 2. (i) *The solution does not approach a cylinder too quickly, in the sense that $\sup_{(\sigma, \tau) \in \Omega_{\text{par}}} |V(\sigma, \tau)| \geq e^{-C\tau}$ for some $C < \infty$.* (ii) *The singularity time $T < \infty$ depends continuously on the initial data g_0 .*

The key consequence of Ansatz Condition 2, which we explain below, is that one eigenmode dominates in the sense that

$$(3.11) \quad V(\sigma, \tau) \approx V_k(\sigma, \tau) := b_k e^{\lambda_k \tau} h_k(\sigma).$$

Because A has pure point spectrum in $L^2(\mathbb{R}, e^{-\sigma^2/4} d\sigma)$, this is reasonable and consistent with Ansatz Condition 1 and with the fact, proved in [2], that singularity formation at the pole is nongeneric in the class of all rotationally symmetric solutions.

In fact, we may assume that equation (3.11) holds for some $k \geq 3$. Here is why. Part (i) of Ansatz Condition 2 implies that some $b_k \neq 0$. (Compare Section 2.16 of [3].) Part (ii) implies that one can without loss of generality choose the free parameter $T = T(g_0)$ so that $b_0 = 0$. (This too is true for the nondegenerate neckpinch; see Section 2.15 of [3].) Ansatz Condition 1 implies that $b_1 = 0$. In Section 6, we discard $k = 2$ as a consequence of Ansatz Condition 4. Hence we have $k \geq 3$, which implies orthogonality to the kernel of A , as promised above.

We can now obtain a rough estimate for the size of the parabolic region. Recall that the main assumption made for the parabolic region, Ansatz Condition 1, applies only so long as $U \approx 1$, hence as long as

$$(3.12) \quad V(\sigma, \tau) = b_k e^{\lambda_k \tau} \sigma^k + \mathcal{O}(e^{\lambda_k \tau} \sigma^{k-2}) \quad (\sigma \rightarrow \infty)$$

is small, hence only as long as $|\sigma| \ll e^{(\frac{1}{2} - \frac{1}{k})\tau}$. We label the region characterized by larger values of $|\sigma|$ the intermediate region.

4. The intermediate region

The assumptions we have made to obtain a formal expansion in the parabolic region are expected to be valid only so long as $V(\sigma, \tau)$ is much smaller than one.

⁵Recall that $b_2(\tau) = (8\tau)^{-1}$ for the rotationally symmetric neckpinch [3].

We thus leave the parabolic region and enter the next region for those values of (σ, τ) for which $V(\sigma, \tau)$ is of order one.

By Ansatz Condition 2 and its consequence (3.11), the dominant term in the Fourier expansion of V is a polynomial with leading term comparable to $e^{\lambda_k \tau} \sigma^k$. So we find it useful to introduce a new space variable

$$(4.1) \quad \rho := (e^{\lambda_k \tau})^{1/k} \sigma = e^{(\frac{1}{k} - \frac{1}{2})\tau} \sigma,$$

and use ρ to demarcate the intermediate ($\rho \approx 1$) region.

In the intermediate region Ω_{int} , we define

$$(4.2) \quad W(\rho, t) = U(\sigma, \tau) \quad (= 1 + V(\sigma, \tau)).$$

Recalling the evolution equation (3.4) satisfied by U , one computes that W satisfies the PDE

$$\begin{aligned} & \frac{1}{2}(W - W^{-1}) - \frac{\rho}{k} W_\rho \\ &= e^{-\tau} W_t - e^{(2/k-1)\tau} \left\{ W_{\rho\rho} + (n-1) \frac{W_\rho^2}{W} - n \int_0^\rho \frac{W_{\rho\rho}}{W} d\rho \right\}. \end{aligned}$$

This appears to be a small time-dependent perturbation of an ODE in ρ . We make this intuition precise in the following additional assumption:

Ansatz Condition 3. W_t , W_ρ , and $W_{\rho\rho}$ are all bounded in the intermediate region.

Ansatz Condition 3 implies that $\frac{\rho}{k} W_\rho - \frac{1}{2} W + \frac{1}{2} W^{-1} = o(1)$ as $\tau \rightarrow \infty$. This suggests that $W(\rho, t) \approx \tilde{W}(\rho, t)$, where \tilde{W} solves the first-order ODE

$$(4.3) \quad \frac{\rho}{k} \tilde{W}_\rho - \frac{1}{2} \tilde{W} + \frac{1}{2} \tilde{W}^{-1} = 0.$$

One readily determines that the general solution to equation (4.3) is

$$(4.4) \quad \tilde{W}(\rho, t) = \sqrt{1 - (\rho/c)^k}.$$

Notice that in solving this ODE, we obtain a “constant of integration” $c(t)$ which may *a priori* depend on time. Matching considerations will show that c is in fact constant. (The minus sign above is chosen so that the singularity occurs at the right pole if $c > 0$.)

In order to match the fields in the intermediate region with those in the parabolic region, it is useful to consider the asymptotic expansion of \tilde{W} about $\rho = 0$; one gets

$$\tilde{W}(\rho, t) = 1 - \frac{1}{2}(\rho/c(t))^k - \frac{1}{8}(\rho/c(t))^{2k} - \dots.$$

Thus for $0 < \rho = (e^{\lambda_k \tau})^{1/k} \sigma \ll 1$, Ansatz Condition 3 implies that

$$(4.5) \quad W(\rho, t) \approx \tilde{W}(\rho, t) = 1 - \frac{c(t)^k}{2} e^{\lambda_k \tau} \sigma^k + \dots,$$

which matches the parabolic expansion (3.12) if $c(t)$ is determined by b_k , which is constant for all $k \geq 3$. It is convenient in what follows to regard b_k as determined by $c > 0$; thus one has

$$(4.6) \quad b_k = -\frac{1}{2} c^{-k}.$$

Going in the other direction, we expect to transition to the tip region where $W \approx 0$, namely where $\rho \approx c$. It is unsurprising that one gets the same conclusion from equation (3.12) by solving $V \approx -1$ for large $|\sigma|$.

5. The outer region

For $\rho \gg 1$, which marks the outer region Ω_{out} , we unwrap definitions (3.2), (3.3), (4.1), and (4.2) to obtain

$$(5.1) \quad \psi(s, t)^2 = 2(n-1)e^{-\tau}W(e^{\tau/k}s, \tau)^2 \approx 2(n-1) \left[(T-t) - \left(\frac{s}{c}\right)^k \right].$$

If $k \geq 3$ is odd, then equation (5.1) implies that the $g(t)$ -measure of the open set $(0, 1) \times \mathbb{S}^n$ goes to zero as $t \nearrow T$. Moreover, the $t = T$ limit profile of ψ can vanish to arbitrarily high order, behaving like $s^{k/2}$ as $s \nearrow 0$.

If $k \geq 4$ is even, then equation (5.1) implies that the solution encounters a global singularity at $t = T$, with the whole manifold shrinking into a non-round point.

In either case, a set of nonzero $g(0)$ -measure is destroyed, in contrast to the nondegenerate Type-I neckpinches studied in [2, 3], which become singular only on the hypersurface $\{x_0\} \times \mathbb{S}^n$. Note that it is possible for local Type-I singularities to destroy sets of nonzero measure. See examples by one of the authors [12], as well as recent work of Enders, Müller, and Topping [11].

6. The tip region

Obtaining precise asymptotics at the right pole is complicated by the fact that the natural geodesic polar coordinate system (2.1) becomes singular there. As in [1], we overcome this difficulty by choosing new local coordinates.

The fact that $\psi_s(s(1, t), t) = -1$ for all $t < T$ implies that $\psi_s < 0$ in a small time-dependent neighborhood of the right pole $x = 1$. So we may regard $\psi(s, t)$ as a new local radial coordinate, thereby regarding s as a function of ψ and t . (Compare [4] and [1].) More precisely, there is a function $y(\psi, t) < 0$ defined for small $\psi > 0$ and times t near T such that

$$(6.1) \quad \psi_s(s, t) = y(\psi(s, t), t).$$

In terms of this coordinate, the metric takes the form

$$(6.2) \quad g = y(\psi(s, t), t)^{-2} (d\psi)^2 + \psi^2 g_{\text{can}},$$

with $y(\psi, t)$ the unknown function whose evolution controls the geometry near the pole.

For convenience, we replace $y < 0$ by the quantity $z = y^2$, which evolves by the PDE $z_t = \mathcal{F}_\psi[z]$, where

$$(6.3) \quad \mathcal{F}_\psi[z] := \frac{1}{\psi^2} \left\{ \psi^2 z z_{\psi\psi} - \frac{1}{2} (\psi z_\psi)^2 + (n-1-z) \psi z_\psi + 2(n-1)(1-z)z \right\}.$$

For the tip region, we now introduce a t -dependent expansion factor for the radial coordinate, setting

$$(6.4) \quad \gamma(s, t) := \Gamma(\tau(t)) \psi(s, t),$$

with Γ to be determined below by matching considerations. Defining

$$(6.5) \quad Z(\gamma, t) := z(\psi, t) \quad (= y^2(\psi, t)),$$

one determines from equation (6.3) that Z satisfies

$$(6.6) \quad \Gamma^{-2}(Z_t + \Gamma^{-1}\Gamma_\tau \gamma Z_\gamma) = \mathcal{F}_\gamma[Z],$$

where $\mathcal{F}_\gamma[\cdot]$ is the operator appearing in equation (6.3), with all ψ derivatives replaced by γ derivatives.

As we observe in Appendix B, the ODE $\mathcal{F}_\gamma[\tilde{Z}] = 0$ admits a one-parameter family of complete solutions satisfying the boundary conditions $\tilde{Z}(0) = 1$ and $\tilde{Z}(\infty) = 0$. Any such solution is given by

$$\tilde{Z}(\gamma) = B\left(\frac{\gamma}{a}\right),$$

where $a > 0$ is an arbitrary scaling parameter, and B is the profile function of the *Bryant steady soliton* metric,

$$g_B = B(r)^{-1}(dr)^2 + r^2 g_{\text{can}},$$

discovered in unpublished work of Bryant, who proved that it is, up to homothety, the unique complete rotationally symmetric non-flat steady gradient soliton on \mathbb{R}^{n+1} for $n \geq 2$. Recent results of Cao and Chen (followed by a simplified proof by Bryant) allow one to replace the words “rotationally symmetric” in the uniqueness statement above by the words “locally conformally flat” [6]. Uniqueness under the assumption of local conformal flatness for $n+1 \geq 4$ was also proved independently by Catino and Mantegazza. Recent progress by Brendle allows one to replace local conformal flatness by the condition that a certain vector field $V := \nabla R + \varrho(R)\nabla f$ decays fast enough at spatial infinity [5]. (Here, f is the soliton potential function, and $\varrho(R)$ is chosen so that V vanishes on the Bryant soliton.) We briefly review some relevant properties of the Bryant soliton in Appendix B.

Numerical studies for Ricci flow of rotationally symmetric neckpinches [13, 14] strongly support the contention that for certain initial data, the flow near the tip approaches a Bryant soliton model. Results of Gu and Zhu also support this expectation [16]. Therefore, in the tip region, we adopt the following assumptions, which suggest that a solution of (6.6) should be approximated by a steady-state solution \tilde{Z} of $\mathcal{F}_\gamma[\tilde{Z}] = 0$, that is, by a suitably scaled Bryant soliton. These assumptions complete our general *Ansatz*.

Ansatz Condition 4. *In the tip region, we assume that the LHS of equation (6.6) is negligible compared to the RHS for $t \approx T$. To wit, we assume that (i) $\Gamma(\tau) \gg e^{\tau/2}$, (ii) $\Gamma_\tau = o(\Gamma^3)$, and (iii) $Z_t = o(\Gamma^2)$, all as $t \nearrow T$.*

As noted above, the choice of the expansion factor $\Gamma(\tau)$ is determined by matching at the intersection of the tip and parabolic regions. We now discuss this determination. Below, we verify that the form of $\Gamma(\tau)$ we obtain in equation (6.11) satisfies the conditions of Ansatz Condition 4 as long as $k \geq 3$.

To study the consequences of matching at the tip-parabolic intersection, we first recall that equation (3.11), with $\sigma = e^{\tau/2}s$, implies that

$$(6.7) \quad \psi(s, t) = \sqrt{2(n-1)}e^{(-1/2+\lambda_k)\tau}[1+o(1)]b_k\sigma^k$$

and

$$(6.8) \quad \psi_s(s, t) = \sqrt{2(n-1)}e^{\lambda_k\tau}[1+o(1)]kb_k\sigma^{k-1}$$

hold for large $|\sigma|$. On the other side of the interface — at the outer boundary of the tip region — it follows from Proposition 1 in Appendix B and the choice ($c_2 = 1$)

of normalization there that

$$(6.9) \quad \psi_s(s, t) \approx \sqrt{\tilde{Z}(\gamma, t)} \approx a\Gamma^{-1}\psi^{-1}(s, t).$$

Comparing (6.7), (6.8), and (6.9) at $\rho = c$, i.e. at $\sigma = e^{-\frac{\lambda_k}{k}\tau}c$, and using (4.6), one finds that the asymptotic expansions match provided that

$$(6.10) \quad a = \frac{k(n-1)}{2c}$$

and

$$(6.11) \quad \Gamma = e^{(1-\frac{1}{k})\tau}.$$

This yields a one-parameter family of formal solutions indexed by the scaling parameter $c > 0$. More precisely, the free parameters a , b_k , and c obtained in the tip, parabolic, and intermediate regions, respectively, are reduced by the tip-parabolic and parabolic-intermediate matching conditions to a single parameter, which we take to be c .

The choice of Γ in equation (6.11) satisfies properties (i) and (ii) of Ansatz Condition 4 provided that $k > 2$. Moreover, the first-order term in our approximate solution is stationary, so that property (iii) is satisfied automatically.

We now recall from Section 2 that an $\text{SO}(n+1)$ -invariant metric has two distinguished sectional curvatures, which we call K and L . In terms of the present set of local coordinates, the sectional curvatures of $g = z^{-1}(\text{d}\psi)^2 + \psi^2 g_{\text{can}}$ are given by $K = -z_\psi/(2\psi)$ and $L = (1-z)/\psi^2$. At the pole $x = 1$, these are equal and easily computed. Again by Proposition 1 in Appendix B, one has

$$(6.12) \quad K|_{x=1} = L|_{x=1} = \lim_{\psi \searrow 0} \frac{1 - Z(\gamma, t)}{\psi^2} = \frac{\Gamma^2}{a^2} = \frac{a^{-2}}{(T-t)^{2-2/k}},$$

where $k \geq 3$. It follows that these formal solutions exhibit the characteristic Type-II curvature blow-up behavior expected of degenerate neckpinch solutions. Notice that the limiting behavior as $k \rightarrow \infty$ matches the $(T-t)^{-2}$ blow-up rate first observed by Daskalopoulos and Hamilton in their rigorous treatment [9] of complete Type-II Ricci flow singularities on \mathbb{R}^2 , where Ricci flow coincides with the logarithmic fast diffusion equation $u_t = \Delta \log u$. The asymptotic profile of its blow-up was derived formally by King [18] and recovered rigorously by Daskalopoulos and Šešum [10].

7. Conclusions

Gu and Zhu [16] prove that Type-II Ricci flow singularities develop from non-generic rotationally symmetric Riemannian metrics on \mathbb{S}^{n+1} ($n \geq 2$), having the form $g = (\text{d}s)^2 + \psi^2(s) g_{\text{can}}$ in local coordinates on $\mathbb{S}^{n+1} \setminus \{P_\pm\}$.

Our work above describes and provides plausibility arguments for a detailed asymptotic profile and rate of curvature blow-up that we predict some (though not necessarily all) such solutions should exhibit. We summarize our prediction as follows.

Conjecture. *For every $n \geq 2$, every $k \geq 3$, and every $c > 0$, there exist Ricci flow solutions $g(t)$ that satisfy the conditions outlined in our Basic Assumption and*

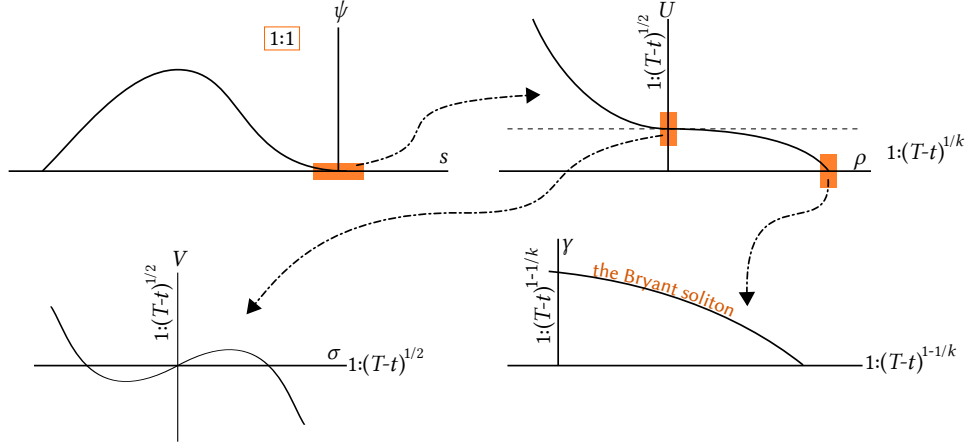


FIGURE 1. An asymmetric degenerate neckpinch viewed at various scales.

develop a degenerate neckpinch singularity at the right pole at some $T < \infty$. The singularity is *Type-II* — slowly forming — with

$$\sup_{x \in \mathbb{S}^{n+1}} |\text{Rm}(x, t)| \sim \frac{C}{(T - t)^{2-2/k}}$$

attained at the pole. Its asymptotic profile is as follows, where $s(x, t)$ represents arc-length with respect to $g(t)$ measured from the location of the smallest nondegenerate neck.

OUTER REGION: As $t \nearrow T$, one has

$$\psi(s, t) = [1 + o(1)] \sqrt{2(n-1) \left[(T-t) - \left(\frac{s}{c} \right)^k \right]}$$

holding for $-\varepsilon \leq s \leq c(T-t)^{1/k}$ if k is odd, and for $|s| \leq c(T-t)^{1/k}$ if k is even.

INTERMEDIATE REGION: As $t \nearrow T$, one has

$$\frac{\psi(s, t)}{\sqrt{2(n-1)(T-t)}} = [1 + o(1)] \sqrt{1 - \frac{(s/c)^k}{T-t}}$$

on an interval $\varepsilon(T-t)^{1/k} \leq s \leq \varepsilon^{-1}(T-t)^{1/k}$.

PARABOLIC REGION: As $t \nearrow T$, one has

$$\frac{\psi(s, t)}{\sqrt{2(n-1)(T-t)}} = 1 - \frac{1 + o(1)}{2c^k} \frac{\sqrt{(T-t)^k}}{T-t} h_k \left(\frac{s}{\sqrt{T-t}} \right)$$

on an interval $\varepsilon\sqrt{T-t} \leq s \leq \varepsilon(T-t)^{1/k}$, where $h_k(\cdot)$ denotes the k^{th} Hermite polynomial, normalized so that its highest-order term has coefficient 1.

TIP REGION: A Bryant soliton forms in a neighborhood of the pole. Specifically, with respect to a rescaled local radial coordinate

$$\gamma(s, t) = \frac{\psi(s, t)}{(T-t)^{1-1/k}}$$

near the right pole, in which the metric takes the form

$$g = Z(\gamma, t)^{-1} (d\psi)^2 + \psi^2 g_{\text{can}},$$

one has

$$Z(\gamma, t) = [1 + o(1)] B \left(\frac{2c\gamma}{k(n-1)} \right) \quad (t \nearrow T),$$

where B denotes the Bryant soliton — up to scaling, the unique complete locally conformally flat⁶ non-flat steady gradient soliton on \mathbb{R}^{n+1} .

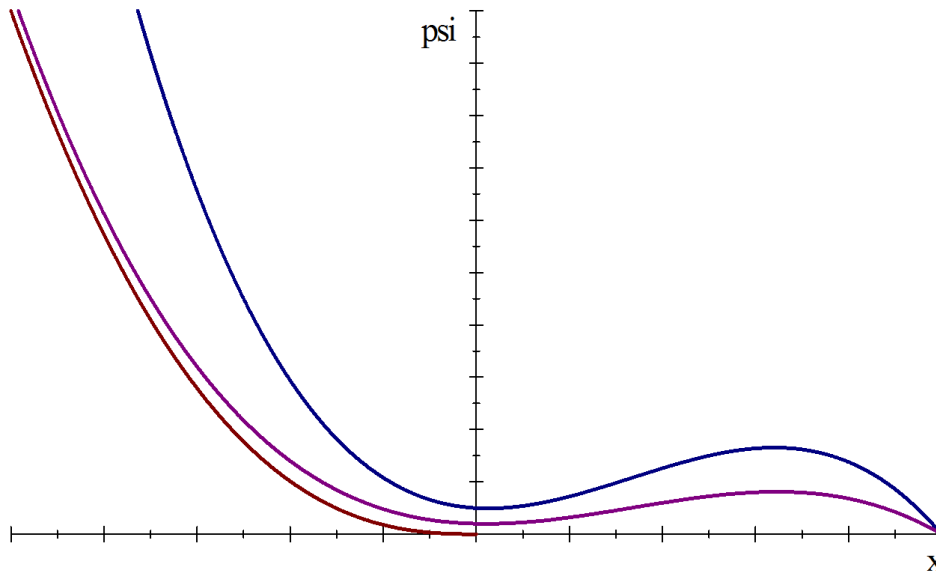


FIGURE 2. An asymmetric degenerate neckpinch viewed without rescaling in the (non-geometric) x coordinate system.

Many aspects of this work are familiar. Indeed, our predicted rate of curvature blowup matches that of the examples of Type-II mean curvature flow singularities rigorously constructed by Velázquez and one of the authors [4]. The “global singularities” encountered by the symmetric (k even) profiles considered here agree with the intuition obtained from such rigorous examples for mean curvature flow. Moreover, the Ricci flow singularities numerically simulated by Garfinkle and another of the authors [13, 14] are qualitatively similar to the case $k = 4$ considered here.

On the other hand, it was perhaps not obvious *a priori* that a Type-II Ricci flow solution would vanish on an open set $(0, 1) \times \mathbb{S}^n$ of the original manifold. This occurs for the asymmetric (k odd) profiles considered here, which correspond to the “intuitive solutions” predicted and sketched by Hamilton [17, Section 3].

Motivated by the fact that conjectures provide direction and structure for the development of rigorous new mathematics, it is our hope that the formal derivations in this paper facilitate further study of Type-II (degenerate) Ricci flow singularity

⁶Uniqueness also holds if local conformal flatness is replaced by a suitable condition at spatial infinity; see [5].

formation. In particular, we intend in forthcoming work to provide a rigorous proof that there exist solutions exhibiting the asymptotic behavior formally described here. Further study is also needed to determine what (if any) other asymptotic behaviors are possible.

Appendix A. Evolution equations in rescaled coordinates

In this appendix, we (partially) explain our choices of space and time dilation in the parabolic region, and we derive the evolution equations satisfied by a rescaled solution.

Let $T < \infty$ denote the singularity time; let

$$(A.1) \quad \tau = -\log(T - t);$$

and let

$$(A.2) \quad \sigma = e^{\beta\tau} s,$$

where β is a constant to be chosen. Define a rescaled solution $U(\sigma, \tau)$ by

$$(A.3) \quad \psi(s, t) = \sqrt{2(n-1)} e^{-\alpha\tau} U(\sigma, \tau),$$

where α is another constant to be determined.

It is straightforward to calculate that

$$\begin{aligned} \psi_t &= \sqrt{2(n-1)} e^{(1-\alpha)\tau} (U_\tau + \sigma_\tau U_\sigma - \alpha U), \\ \psi_s &= \sqrt{2(n-1)} e^{(\beta-\alpha)\tau} U_\sigma, \\ \psi_{ss} &= \sqrt{2(n-1)} e^{(2\beta-\alpha)\tau} U_{\sigma\sigma}. \end{aligned}$$

The factor σ_τ is necessary for σ and τ to be commuting variables. It is given by

$$(A.4) \quad \sigma_\tau = \beta\sigma + e^{-\tau/2} \frac{\partial s}{\partial t} = \beta\sigma + nI,$$

where I is the non-local term

$$(A.5) \quad I := \int_0^\sigma \frac{U_{\sigma\sigma}}{U} d\sigma.$$

It follows from equation (2.6b) that U satisfies

$$(A.6) \quad e^{(1-2\beta)\tau} (U_\tau + \sigma_\tau U_\sigma - \alpha U) = U_{\sigma\sigma} + (n-1) \frac{U_\sigma^2}{U} - e^{2(\alpha-\beta)\tau} \frac{1}{2U}.$$

If $\beta < 1/2$, then for $\tau \gg 0$, U should approximate a translating solution of a first-order equation. If $\beta > 1/2$, then for $\tau \gg 0$, U should approximate a stationary solution of an elliptic equation. The choice $\beta = 1/2$ is thus necessary if one expects U to be modeled by the solution of a parabolic equation.⁷

Now suppose that $\beta = 1/2$ and write $U = 1 + V$. It follows from the considerations above that V evolves by

$$(A.7) \quad V_\tau = V_{\sigma\sigma} - \left(\frac{\sigma}{2} + nI \right) V_\sigma + (n-1) \frac{V_\sigma^2}{1+V} + \alpha(1+V) - \frac{e^{(2\alpha-1)\tau}}{2(1+V)}.$$

⁷Caveat: it is not necessarily the case that a parabolic equation will dominate in the “parabolic region;” for example, see the degenerate singularity considered in [1].

Linearizing about $V = 0$, i.e. $U = 1$, one finds that $V_\tau = \tilde{A}V + \mathcal{O}(V^2)$, where

$$\tilde{A} : V \mapsto V_{\sigma\sigma} - \frac{\sigma}{2}V_\sigma + \left[\alpha + \frac{1}{2}e^{(2\alpha-1)\tau} \right] V + \left[\alpha - \frac{1}{2}e^{(2\alpha-1)\tau} \right] V.$$

The choice $\alpha = 1/2$ thus results in \tilde{A} becoming the autonomous linear operator

$$(A.8) \quad A : V \mapsto V_{\sigma\sigma} - \frac{\sigma}{2}V_\sigma + V.$$

Appendix B. The Bryant Soliton

The Bryant soliton, discovered in unpublished work of Robert Bryant, is up to homothetic scaling, the unique complete non-flat locally conformally flat steady gradient soliton on \mathbb{R}^{n+1} for $n \geq 2$. (See [6] and [7].) Uniqueness also holds under the assumption that a vector field $V := \nabla R + \varrho(R)\nabla f$ decays fast enough at spatial infinity. (Recall that f is the soliton potential function, and $\varrho(R)$ is chosen so that V vanishes on the Bryant soliton; see [5].)

Numerical simulations by Garfinkle and one of the authors suggest that a degenerate neckpinch solution should converge to the Bryant soliton after rescaling near the north pole [13, 14]. Results of Gu and Zhu also support this expectation [16].

Here we recall some relevant properties of these solutions. It is convenient to consider a one-parameter family of Bryant soliton profile functions $B(\cdot)$ depending on a positive parameter that encodes the scaling invariance mentioned above. For more information about the Bryant soliton (including proofs of the following claims), see Appendix C of [1] and Chapter 1, Section 4 of [8].

Proposition 1 (Properties of the Bryant soliton profile function).

- (1) The ODE $\mathcal{F}_r[z] = 0$, where

$$\mathcal{F}_r[z] := \frac{1}{r^2} \left\{ r^2 z z_{rr} - \frac{1}{2} (r z_r)^2 + (n-1-z) r z_r + 2(n-1)(1-z)z \right\},$$

admits a unique one-parameter family of complete solutions satisfying $Z(0) = 1$ and $Z(\infty) = 0$. These are given by

$$Z(r) = B\left(\frac{r}{\varrho}\right)$$

for $\varrho > 0$, where B is the Bryant soliton profile function. Each member of the one-parameter family of complete smooth metrics given by

$$g = Z^{-1}(r)(dr)^2 + r^2 g_{\text{can}},$$

is called a Bryant soliton.

- (2) $B(r)$ is strictly monotone decreasing for all $r > 0$.
 (3) Near $r = 0$, B is smooth and has the asymptotic expansion

$$B(r) = 1 + b_2 r^2 + \frac{n}{n+3} b_2^2 r^4 + \frac{n(n-1)}{(n+3)(n+5)} b_2^3 r^6 + \cdots,$$

where $b_2 < 0$ is arbitrary.

(4) Near $r = +\infty$, B is smooth and has the asymptotic expansion

$$B(r) = c_2 r^{-2} + \frac{4-n}{n-1} c_2^2 r^{-4} + \frac{(n-4)(n-7)}{(n-1)^2} c_2^3 r^{-6} + \cdots,$$

where $c_2 > 0$ is arbitrary.

The arbitrariness of b_2 and c_2 encodes the scaling invariance of the Bryant soliton. In this paper, we fix $c_2 = 1$ in order to make explicit the dependence on the scaling parameter $\varrho > 0$ when writing $Z(r) = B(r/\varrho)$.

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